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STUDY OF THE AERODYNAMIC FORCES AND MOMENTS
ON BODIES OF REVOLUTION IN COMBINED
PITCHING AND YAWING MOTIONS

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ERRATA

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The questions raised in this paper concerning the proper method of writing aerodynamic forces and moments for combined motions have been reconsidered. Analysis based on potential flow theory reveals that the point of view taken in the first part of the report (pp. 7-10) and in the concluding section (pp. 17, 18) is incorrect; the accepted method which was disputed by the authors (pp. 6, 7) is in fact correct.

Consider the test case, steady circular motion, $\sigma = \text{constant}$, $\lambda = \text{constant}$. The argument contained in the report, wherein it is asserted that aerodynamic acceleration terms should vanish for this motion, is based implicitly on the incorrect assumption that the steady-state potential equation referred to rotating coordinates fixed in the body retains the same form it has for steady flow in nonrotating coordinates. That is, it was assumed that the differential equation for steady-state perturbation potential in the body-fixed cylindrical coordinates x, r, μ is

$$-\beta^2 \phi_{xx} + \phi_{rr} + \phi_r/r + \phi_{\mu\mu}/r^2 = 0$$

in which case there is no possibility for the existence of acceleration forces and moments. Closer analysis shows, however, that the potential equation becomes

$$-\beta^2 \phi_{xx} + \phi_{rr} + \phi_r/r + \phi_{\mu\mu}/r^2 = -(2M^2/V) \dot{\phi} \phi_{x\mu}$$

where $\dot{\phi}$ is the angular velocity of the body-fixed coordinate system about the axis of symmetry. The changed form of the potential equation must be interpreted as meaning that even though the flow appears steady to an observer in the body-fixed coordinate system, he is still able to discern that the body is turning. This fact invalidates the argument that aerodynamic acceleration terms are necessarily absent when the flow is steady. For unsteady flow relative to body-fixed coordinates the perturbation potential equation becomes

$$-\beta^2 \phi_{xx} + \phi_{rr} + \frac{\phi_r}{r} + \frac{\phi_{\mu\mu}}{r^2} - \frac{2M^2}{V} \phi_{xt} - \frac{M^2}{V^2} \phi_{tt} = - \frac{2M^2}{V^2} \phi \frac{\partial}{\partial \mu} (V\phi_x + \phi_t)$$

The solution to this equation, subject to the appropriate boundary conditions, has been obtained and confirms the form of the results in current usage (eqs. (11) and (12)).

The section entitled "Magnus Forces" (pp. 10-12) remains correct and the section entitled "Significance of New Formulation" (pp. 12-17) can be made correct if the substitution

$$C_{mp\alpha} = C_{mp\alpha} - C_{m\dot{\alpha}}$$

is made in equation (31) and thereafter.

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PITCHING AND YAWING MOTIONS

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SUMMARY

Significant errors are pointed out in the accepted practice of writing the aerodynamic forces and moments that act on a body of revolution during pitching and yawing motion. A more correct formulation is presented which shows that, even with linear aerodynamic coefficients, the differential equations governing the motion are fundamentally nonlinear. This permits an explanation of certain types of motion which have been observed experimentally and which have been explainable previously only with the assumption of nonlinear aerodynamic characteristics.

INTRODUCTION

As the performance and appearance of modern aircraft, missiles, and rockets have come to resemble those of projectiles, it has been recognized that there is a need for a unified theory of motion which includes in a common language those special features of projectile motion familiar to ballisticians and those features of airplane motion familiar to aerodynamicists. Several researchers have sought to fulfill this need, the most noteworthy contributions being those of Nicolaides (ref. 1) and Charters (ref. 2).

While the kinematic aspects of the problem have been successfully united in these contributions, the aerodynamic aspects still reflect the prior interests of aerodynamicists accustomed to the motions of winged vehicles. Winged vehicles, of course, generally have preferred modes of transverse motion which deviate only slightly from fixed reference planes, and this has led the aerodynamicist to view those motions as projections in such planes. This concept has been carried over into the unified theory and the purpose here will be to show that it leads to errors when used to represent the aerodynamic forces and moments acting on a vehicle that has no preferred plane of transverse motion. Then it will be shown by adapting an older concept, namely the use of planes that rotate with the resultant angle-of-attack vector (refs. 3 and 4), that these errors can be avoided and a more correct representation of the forces and moments can be easily

formulated. Finally, results of the revised formulation will be incorporated in the equations governing the motion of bodies of revolution and a discussion will be given of the types of motion made explainable by the changes.

SYMBOLS

C_m	pitching-moment coefficient, $\frac{\text{pitching moment}}{QS\bar{l}}$
C_{mab}	rate of change of Magnus pitching-moment coefficient about an axis in the plane of b with respect to a and b ; $\left(\frac{\partial^2 C_m}{\partial a \partial b}\right)_{\substack{a \rightarrow 0 \\ b \rightarrow 0}}; a = \frac{Pl}{V}, \frac{p\bar{l}}{V}; b = \alpha, \frac{q\bar{l}}{V}, \frac{\dot{\alpha}\bar{l}}{V}$
C_N	normal-force coefficient, $\frac{\text{normal force}}{QS}$
C_{Nab}	rate of change of Magnus normal-force coefficient normal to the plane of b with respect to a and b ; $\left(\frac{\partial^2 C_N}{\partial a \partial b}\right)_{\substack{a \rightarrow 0 \\ b \rightarrow 0}}; a = \frac{Pl}{V}, \frac{p\bar{l}}{V}; b = \alpha, \frac{q\bar{l}}{V}, \frac{\dot{\alpha}\bar{l}}{V}$
C_{Nq}, C_{mq}	rate of change of normal-force and pitching-moment coefficients with pitching velocity parameter $\frac{q\bar{l}}{V}$; $\left(\frac{\partial C_N}{\partial q\bar{l}/V}\right)_{q \rightarrow 0}, \left(\frac{\partial C_m}{\partial q\bar{l}/V}\right)_{q \rightarrow 0}$
$C_{N\alpha}, C_{m\alpha}$	rate of change of normal-force and pitching-moment coefficients with angle of attack; $\left(\frac{\partial C_N}{\partial \alpha}\right)_{\alpha \rightarrow 0}, \left(\frac{\partial C_m}{\partial \alpha}\right)_{\alpha \rightarrow 0}$
$C_{N\dot{\alpha}}, C_{m\dot{\alpha}}$	rate of change of normal-force and pitching-moment coefficients with time rate of change of angle-of-attack parameter $\frac{\dot{\alpha}\bar{l}}{V}$; $\left(\frac{\partial C_N}{\partial \dot{\alpha}\bar{l}/V}\right)_{\dot{\alpha} \rightarrow 0}, \left(\frac{\partial C_m}{\partial \dot{\alpha}\bar{l}/V}\right)_{\dot{\alpha} \rightarrow 0}$
$C_N(\sigma), C_m(\sigma)$	normal-force coefficient in the plane of σ and pitching-moment coefficient about an axis normal to the plane of σ
$C_N(\perp\sigma), C_m(\perp\sigma)$	normal-force coefficient in a plane normal to the plane of σ and pitching-moment coefficient about an axis in the plane of σ

C_Y	side-force coefficient, $\frac{\text{side force}}{QS}$
C_Z	$-C_N$
I	moment of inertia about an axis normal to the axis of symmetry and passing through the center of gravity
k	aerodynamic coupling term (eq. (34))
l	reference length
p, q, r	components of angular velocity about x, y, z axes
P	effective spin rate, $\dot{\phi} - \dot{\lambda}$
Q	dynamic pressure, $\frac{1}{2} \rho V^2$
S	reference area
t	time
u, v, w	components of velocity vector V along x, y, z
V	flight velocity
W	velocity normal to body axis of symmetry
x, y, z	nonrolling orthogonal axes with origin at body center of gravity (fig. 1)
X, Y, Z	orthogonal axes with directions fixed in space and origin at body center of gravity (fig. 1)
α	angle of attack (eq. (1))
β	angle of sideslip (eq. (1))
ζ	damping ratio (eq. (30))
θ	angle of pitch (fig. 1)
λ	angular displacement of plane containing σ from xy plane (fig. 1)
ρ	air density
σ	resultant angle of attack (fig. 1)
φ	roll angle (fig. 1)

ψ	angle of yaw (fig. 1)
ω	circular frequency
ω_n	undamped natural circular frequency (eq. (30))
$(\dot{}), (\ddot{})$	$\frac{d()}{dt} \quad \frac{d^2()}{dt^2}$
$()_i$	initial value
$()_a$	asymptotic value

ANALYSIS

The purpose of the following analysis is to show that an error exists in the accepted practice of writing the aerodynamic forces and moments that act on a body of revolution during pitching and yawing motion. To show this clearly, it is desirable to eliminate as many extraneous considerations as possible. Thus, the following conditions are imposed:

- (1) The vehicle's center of gravity traverses a straight path.
- (2) The vehicle's forward velocity, as measured at its center of gravity, is constant in magnitude.
- (3) The vehicle's effective rolling rate is small enough that Magnus forces and moments may be considered to be negligible. (This condition is relaxed in a later section.)

These conditions are easy to visualize if the vehicle is considered to be a wind-tunnel model mounted on a bearing which permits the model to pitch and yaw, and to roll at a slow rate.

Coordinate System

Figure 1 shows the system of axes generally used in the analysis of bodies with rotational symmetry. Two sets of orthogonal axes are shown whose origins lie at the center of gravity: The XYZ axes have fixed directions in space while the xyz axes pitch and yaw with the body through the angles θ and ψ . The y axis is constrained to lie in the XY plane, however, so that the angular position of the body in space is not completely specified. This is permissible for bodies with sufficient roll symmetry because the angle ϕ affects the aerodynamic forces acting on the body only in its time rate of change.

The velocity vector orientation angles α and β , defined according to standard NASA notation as

$$\tan \alpha = \frac{W}{u}, \quad \sin \beta = \frac{V}{V} \quad (1)$$

are equal to the space orientation angles θ and $-\psi$ for the case under consideration for which the path of the center of gravity is a straight line. Two other orientation angles will also be used; the resultant angle of attack σ , and the angle λ between the plane containing σ and the xy plane. These angles are defined as

$$\left. \begin{aligned} \sin \sigma &= \frac{\sqrt{W^2 + V^2}}{V} \\ \tan \lambda &= \frac{W}{V} \end{aligned} \right\} \quad (2)$$

where it is to be noted that σ is taken as always positive. The velocity components are

$$\left. \begin{aligned} u &= V \cos \alpha \cos \beta = V \cos \sigma \\ v &= V \sin \beta = V \sin \sigma \cos \lambda \\ w &= V \sin \alpha \cos \beta = V \sin \sigma \sin \lambda \end{aligned} \right\} \quad (3)$$

Then

$$\left. \begin{aligned} \sin^2 \sigma &= \sin^2 \beta + \sin^2 \alpha \cos^2 \beta \\ \tan \lambda &= \sin \alpha \cot \beta \\ \tan \alpha &= \tan \sigma \sin \lambda \\ \sin \beta &= \sin \sigma \cos \lambda \end{aligned} \right\} \quad (4)$$

which reduce, for small values of α and β , to

$$\left. \begin{aligned} \sigma &= \sqrt{\alpha^2 + \beta^2} \\ \lambda &= \tan^{-1} \left(\frac{\alpha}{\beta} \right) \\ \alpha &= \sigma \sin \lambda \\ \beta &= \sigma \cos \lambda \end{aligned} \right\} \quad (5)$$

Finally, the components of angular velocity about the xyz axes are given by

$$\left. \begin{aligned} p &= \dot{\phi} - \dot{\psi} \sin \theta \\ q &= \dot{\theta} \\ r &= \dot{\psi} \cos \theta \end{aligned} \right\} \quad (6)$$

which reduce, for small angles, to

$$\left. \begin{aligned} p &= \dot{\phi} - \theta r \\ q &= \dot{\theta} \\ r &= \dot{\psi} \end{aligned} \right\} \quad (7)$$

Customary Practice

First, the customary method of writing the forces and moments corresponding to a pitching and yawing motion will be reviewed to show in what sense it is incorrect. Consider that the body executes a motion about its center of gravity consisting of arbitrary variations in both α and β . Then the customary procedure is first to observe the motion as it appears in the plane containing α and to write the forces and moments corresponding to that motion as though it alone existed. Thus, by analogy with a planar motion one would write for C_Z

$$C_Z = -\alpha C_{N\alpha} - \frac{\dot{\alpha} l}{V} (C_{Nq} + C_{N\dot{\alpha}}) \quad (8)$$

Likewise, observing the motion in the β plane, and using the properties of symmetry for a body of revolution, one would write

$$C_Y = -\beta C_{N\alpha} - \frac{\dot{\beta} l}{V} (C_{Nq} + C_{N\dot{\alpha}}) \quad (9)$$

For use in equations of motion it is convenient to transfer these coefficients to axes in and normal to the plane of the resultant angle of attack. The transformation equations are

$$\left. \begin{aligned} C_N(\sigma) &= -C_Z \sin \lambda - C_Y \cos \lambda \\ C_N(l\sigma) &= -C_Z \cos \lambda + C_Y \sin \lambda \end{aligned} \right\} \quad (10)$$

Inserting equations (8) and (9) in (10) and using the relations (5) yields

$$\left. \begin{aligned} C_N(\sigma) &= \sigma C_{N\alpha} + \frac{\dot{\sigma} l}{V} (C_{Nq} + C_{N\dot{\alpha}}) \\ C_N(\perp \sigma) &= \sigma \frac{\dot{\lambda} l}{V} (C_{Nq} + C_{N\dot{\alpha}}) \end{aligned} \right\} \quad (11)$$

A similar procedure for the pitching moment yields

$$\left. \begin{aligned} C_m(\sigma) &= \sigma C_{m\alpha} + \frac{\dot{\sigma} l}{V} (C_{mq} + C_{m\dot{\alpha}}) \\ C_m(\perp \sigma) &= \sigma \frac{\dot{\lambda} l}{V} (C_{mq} + C_{m\dot{\alpha}}) \end{aligned} \right\} \quad (12)$$

It may be verified by referring, for example, to references 1, 2, and 5 that equations (11) and (12) are indeed the force and moment coefficients that would be used in the equations of motion appropriate to the specified conditions (uniform rectilinear motion, Magnus forces neglected).

Now, to see that equations (11) and (12) are incorrect, consider the case of purely circular motion, $\sigma = \text{constant}$. To demonstrate the point completely, let us also specify that $\dot{\lambda}$ be constant and that the body spin rate $\dot{\phi}$ equal $\dot{\lambda}$. The latter specification, $\dot{\phi} = \dot{\lambda}$, ensures that the body does not spin with respect to the stream crossflow velocity, so that Magnus forces, rather than being small, can be said to be identically zero.¹ Then, with both σ and $\dot{\lambda}$ constant, a completely steady motion exists in the sense that every point on the body experiences a normal flow velocity that is invariant with time. Because the stability derivatives $C_{N\dot{\alpha}}$ and $C_{m\dot{\alpha}}$ are by definition the responses to an accelerated uniform normal flow, it is recognized that for steady circular motion all evidence of these terms must vanish in equations (11) and (12). Observe that $C_{N\dot{\alpha}}$ and $C_{m\dot{\alpha}}$ vanish from the expressions in the plane of the resultant angle of attack since $\dot{\sigma} = 0$ but they do not vanish from the expressions for force and moment normal to the plane of the resultant angle of attack. Hence, equations (11) and (12) cannot be correct.

New Formulation

Having shown that the customary method of writing the aerodynamic forces and moments yields incorrect results in at least one limiting case,

¹The accepted practice is incorrect on this point also. See section entitled, "Magnus Forces."

let us now investigate the procedure more closely. Let σ and λ vary arbitrarily with time and consider the velocities normal to the body axis of symmetry induced by these variations. In the σ plane the normal velocity is composed of two components; one due to the resultant angle of attack, $V\sigma$, and one due to the rotation of the body about an axis normal to the σ plane, $-\dot{\sigma}x$. Hence, at body station x in the σ plane

$$\frac{W(\sigma)}{V} = \sigma - \frac{\dot{\sigma}x}{V} \quad (13)$$

The dimensionless acceleration of the normal flow is therefore

$$\frac{\dot{W}(\sigma)}{V^2} = \frac{\dot{\sigma}l}{V} - \frac{\ddot{\sigma}xl}{V^2} \quad (14)$$

In equation (14) the term $\ddot{\sigma}xl/V^2$ may be discarded since it yields a force proportional to pitching acceleration ($C_{N\dot{q}}$) which is in phase with and usually negligible compared to inertial terms in the equations of motion. The term $\dot{\sigma}l/V$, however, is of the same order as $\dot{\sigma}x/V$, and must be retained; observe that $\dot{\sigma}l/V$ is uniform along the body x axis and hence will lead to a normal force proportional to $C_{N\dot{\alpha}}$. The term $\dot{\sigma}x/V$, on the other hand, leads to a normal force proportional to $C_{N\dot{\alpha}}$. As a result of these flows, there is a normal-force coefficient in the plane of σ , $C_N(\sigma)$, and a pitching-moment coefficient about an axis normal to the plane of σ , $C_m(\sigma)$, which can be written as

$$C_N(\sigma) = \sigma C_{N\alpha} + \frac{\dot{\sigma}l}{V} (C_{Nq} + C_{N\dot{\alpha}}) \quad (15a)$$

$$C_m(\sigma) = \sigma C_{m\alpha} + \frac{\dot{\sigma}l}{V} (C_{mq} + C_{m\dot{\alpha}}) \quad (15b)$$

Now consider the flow in the direction normal to the σ plane. The velocity normal to the body axis of symmetry is²

$$\frac{W(\perp\sigma)}{V} = x \frac{\dot{\lambda}}{V} \tan \sigma \approx \sigma x \frac{\dot{\lambda}}{V} \quad (16)$$

This normal flow varies linearly with x , and hence can only yield a force proportional to C_{Nq} . The normal-force coefficient normal to the σ plane and the pitching-moment coefficient about an axis in the σ plane are therefore

²There is ignored here a small normal flow variation across the body diameter. The quantity neglected is, however, antisymmetric with respect to the diameter and hence should yield neither a force normal to the σ plane nor a pitching moment about an axis in the σ plane.

$$C_N(l\sigma) = \sigma \frac{\dot{\lambda}l}{V} C_{Nq} \quad (17a)$$

$$C_m(l\sigma) = \sigma \frac{\dot{\lambda}l}{V} C_{mq} \quad (17b)$$

Equations (15) and (17) are the new results, which are to be compared with the previous results, equations (11) and (12). It is immediately evident from inspection of equations (15) and (17) that, unlike the previous results, they reduce properly for circular motion since $C_{N\dot{\alpha}}$ and $C_{m\dot{\alpha}}$ do not appear in the expressions for force and moment normal to the σ plane.

Let us now attempt to discover where the error occurs in the previous analysis. First, transfer the new results for the force coefficients, equations (15a) and (17a) to the α plane by means of

$$C_Z = -C_N(\sigma) \sin \lambda - C_N(l\sigma) \cos \lambda \quad (18)$$

Inserting equations (15a) and (17a) in (18) gives

$$C_Z = -\sigma \sin \lambda C_{N\alpha} - \frac{\dot{\sigma}l}{V} \sin \lambda (C_{Nq} + C_{N\dot{\alpha}}) - \sigma \frac{\dot{\lambda}l}{V} \cos \lambda C_{Nq} \quad (19)$$

which becomes, by use of equations (5),

$$C_Z = -\alpha C_{N\alpha} - \frac{\dot{\alpha}l}{V} C_{Nq} - \frac{l}{V} \left(\frac{\alpha^2 \dot{\alpha} + \alpha \beta \dot{\beta}}{\alpha^2 + \beta^2} \right) C_{N\dot{\alpha}} \quad (20)$$

On comparison of this result with the corresponding one obtained earlier, equation (8), it is seen that the results agree in the terms containing $C_{N\alpha}$ and C_{Nq} , but disagree in the term containing $C_{N\dot{\alpha}}$. Therefore, consider the component of normal flow velocity in the α plane that leads to a uniform normal acceleration. This component is due to the uniform normal velocity σV in the σ plane. It is

$$\frac{W(\alpha)}{V} = \sigma \sin \lambda \quad (21)$$

Hence the uniform normal acceleration in the α plane is

$$\frac{\dot{W}(\alpha)}{V} = \dot{\sigma} \sin \lambda + \sigma \dot{\lambda} \cos \lambda = \dot{\alpha} \quad (22)$$

Examine the significance of the second term, $\sigma \dot{\lambda} \cos \lambda$. It accounts for the fact that the σ plane is turning at a rate $\dot{\lambda}$. Discounting interference effects, however, the fact that the σ plane turns has no bearing on the force in the σ plane, and likewise must have no bearing on the projection of that force in the α plane, other than through the factor, $\sin \lambda$. That is, the acceleration force in the α plane is $-(\dot{\sigma}l/V) \sin \lambda C_{N\alpha}$, as given in equations (19) and (20), not $-(\dot{\sigma}l/V) C_{N\alpha}$ as given by equation (8). The error in equation (8) is introduced when one fails to recognize that part of the normal acceleration as viewed in the α plane is not involved in the development of an aerodynamic force. Again consider steady circular motion, $\sigma = \text{constant}$, $\dot{\phi} = \dot{\lambda} = \text{constant}$. There is still a uniform normal acceleration in the α plane, now due entirely to the turning of the σ plane. The flow at the body is actually completely steady, however, so that the $C_{N\alpha}$ contribution to the total force coefficient must be zero. It is clear, therefore, that no force contribution should be attributed to the term $\sigma \dot{\lambda} \cos \lambda$.

It remains to discuss a point which has not been considered, either in the former method or the present development. In both methods it is assumed that the forces in a plane can be written as though no motion exists in another plane; that is, that no interactions occur between flow fields. There is no a priori reason for assuming this; the hope is, of course, that such interactions are negligibly small. It should be noted that the interactions referred to need not entirely depend for their existence on the presence of viscous effects. In fact, components due to potential flow may exist; if so, they can be analyzed and such an analysis may lead to useful results.³ In a later section a simple experiment is also suggested for investigating the magnitude of interactions.

Magnus Forces

The accepted practice of writing the Magnus forces can be criticized on two counts: First, it is subject to the same sort of error already shown to exist in the forces due to the normal flow; that is, by considering projections of the motion as they appear in fixed planes one inadvertently introduces a spurious acceleration force. Second, in the accepted practice it is said that Magnus forces are identically zero when p , the component of the vehicle's angular velocity about the axis of symmetry, is zero. But now consider again circular motion in which the vehicle spins at a rate $\dot{\phi} = \dot{\lambda}$. The angular velocity p is not necessarily zero, but Magnus forces are identically zero because the vehicle does not spin with respect to the crossflow velocity. The point is, of

³For example, with the use of steady potential flow theory, the antisymmetric component of flow normal to the σ plane mentioned in footnote 2 can be shown to cause a small force to exist in the σ plane.

course, that it is the spin rate relative to the crossflow, $\dot{\phi} - \dot{\lambda}$, rather than p , that is the significant parameter in determining the Magnus forces.

The way to avoid the errors is therefore clear: The first error is avoided by writing the forces directly in the plane of and normal to the plane of the resultant angle of attack. The second error is avoided by recognizing that Magnus forces are generated by the effective rotation of the body with respect to the crossflow velocity, that is, by use of $\dot{\phi} - \dot{\lambda}$ instead of p . This procedure gives for the Magnus force coefficients which are to be added to equations (15a) and (17a)

$$\left. \begin{aligned} C_N(\sigma) &= -\sigma \frac{\dot{\lambda}l}{V} \left(\frac{\dot{\phi}l}{V} - \frac{\dot{\lambda}l}{V} \right) C_{NPq} \\ C_N(\perp\sigma) &= \sigma \left(\frac{\dot{\phi}l}{V} - \frac{\dot{\lambda}l}{V} \right) C_{NP\alpha} + \frac{\dot{\sigma}l}{V} \left(\frac{\dot{\phi}l}{V} - \frac{\dot{\lambda}l}{V} \right) (C_{NPq} + C_{NP\dot{\alpha}}) \end{aligned} \right\} \quad (23)$$

Similarly, the Magnus moment coefficients to be added to equations (15b) and (17b) become

$$\left. \begin{aligned} C_m(\sigma) &= -\sigma \frac{\dot{\lambda}l}{V} \left(\frac{\dot{\phi}l}{V} - \frac{\dot{\lambda}l}{V} \right) C_{mPq} \\ C_m(\perp\sigma) &= \sigma \left(\frac{\dot{\phi}l}{V} - \frac{\dot{\lambda}l}{V} \right) C_{mP\alpha} + \frac{\dot{\sigma}l}{V} \left(\frac{\dot{\phi}l}{V} - \frac{\dot{\lambda}l}{V} \right) (C_{mPq} + C_{mP\dot{\alpha}}) \end{aligned} \right\} \quad (24)$$

The subscript P rather than p is used in the Magnus coefficients to indicate clearly that the derivative is to be taken with respect to the effective spin rate, $\dot{\phi} - \dot{\lambda}$. Thus, $P \equiv \dot{\phi} - \dot{\lambda}$, and, for example

$$C_{mP\alpha} \equiv \frac{\partial^2 C_m}{\partial \left(\frac{\dot{\phi}l}{V} - \frac{\dot{\lambda}l}{V} \right) \partial \alpha}; \quad C_{mPq} \equiv \frac{\partial^2 C_m}{\partial \left(\frac{\dot{\phi}l}{V} - \frac{\dot{\lambda}l}{V} \right) \partial \left(\frac{ql}{V} \right)} \quad (25)$$

Now, compare equations (24) for the Magnus moment coefficients with the expressions used in current practice (cf. ref. 2, for rectilinear motion). The latter expressions are

$$\left. \begin{aligned} C_m(\sigma) &= -\sigma \frac{\dot{\lambda}l}{V} \frac{pl}{V} (C_{mpq} + C_{mp\dot{\alpha}}) \\ C_m(\perp\sigma) &= \sigma \frac{pl}{V} C_{mp\alpha} + \frac{\dot{\sigma}l}{V} \frac{pl}{V} (C_{mpq} + C_{mp\dot{\alpha}}) \end{aligned} \right\} \quad (26)$$

It is seen that equation (26) has an extraneous term $C_{m\dot{p}\dot{\alpha}}$ in $C_m(\sigma)$. The more significant difference, however, is the use of $P (= \dot{\phi} - \dot{\lambda})$ in place of p . The difference can be made explicit by writing P for small angles as

$$P = p - \frac{1}{\sigma^2} \left[\dot{\alpha}\beta - \alpha\dot{\beta}(1 - \sigma^2) \right] \quad (27)$$

For rapid spin rates, such as those for spinning shells, the second term can be negligibly small in comparison with p . Also, for planar motion in either the α or the β plane the second term is identically zero. Hence, in these three cases, no significant error is introduced by the use of p rather than P for the spin rate. In general, however, the second term can be as large or larger than p , and its neglect would be justifiable in individual cases only after a careful analysis of relative orders of magnitude.

DISCUSSION

Significance of New Formulation

In order to examine the significance of the analysis in the preceding sections, consider now the motion of a nonspinning body which is free to pitch and yaw. By a nonspinning body is meant here that $\dot{\phi} = 0$. As usually written, the equations of motion in the coordinates α and β would then be

$$\left. \begin{aligned} I\ddot{\alpha} &= QSl \left[\alpha C_{m\alpha} + \frac{\dot{\alpha}l}{V} (C_{mq} + C_{m\dot{\alpha}}) \right] \\ I\ddot{\beta} &= QSl \left[\beta C_{m\alpha} + \frac{\dot{\beta}l}{V} (C_{mq} + C_{m\dot{\alpha}}) \right] \end{aligned} \right\} \quad (28)$$

where the equations have been linearized on the basis that $\alpha, \beta, \dot{\alpha}, \dot{\beta}$ are small and that their squares and products may be neglected. These equations transform, in σ, λ coordinates, to

$$\left. \begin{aligned} I(\ddot{\sigma} - \sigma\dot{\lambda}^2) &= QSl \left[\sigma C_{m\alpha} + \frac{\dot{\sigma}l}{V} (C_{mq} + C_{m\dot{\alpha}}) \right] \\ I(\sigma\ddot{\lambda} + 2\dot{\sigma}\dot{\lambda}) &= QSl \sigma \frac{\dot{\lambda}l}{V} (C_{mq} + C_{m\dot{\alpha}}) \end{aligned} \right\} \quad (29)$$

The solution of equations (28) is

$$\epsilon = \epsilon_0 e^{-\xi \omega_n t} \sin(\omega t + \gamma_\epsilon) \quad (30)$$

where

$$\epsilon = \alpha \text{ or } \beta$$

$$\epsilon_0 = \sqrt{\epsilon_i^2 + \frac{1}{\omega^2} (\dot{\epsilon}_i - \xi \omega_n \epsilon_i)^2}$$

$$\omega_n^2 = - \frac{C_{m\alpha}}{I} Q S l$$

$$\xi \omega_n = - \frac{(C_{mq} + C_{m\dot{\alpha}})}{I} \frac{Q S l^2}{2V}$$

$$\gamma_\epsilon = \tan^{-1} \left(\frac{\omega \epsilon_i}{\dot{\epsilon}_i - \xi \omega_n \epsilon_i} \right)$$

$$\omega = \omega_n \sqrt{1 - \xi^2}$$

As a result of the harmonic solution (30) for both α and β , the motion when plotted in α, β coordinates consists of lines, circles, or ellipses, dependent only on the initial conditions. Since the α and β motions have identical periods and damping factors, the type of motion determined by the initial conditions (lines, circles, or ellipses) is incapable of degenerating into either of the other two types, but must exist for all finite time.

Now consider the equations of motion in the σ, λ coordinates corresponding to the new formulation. The inertia terms given in equations (29) will be retained; the aerodynamic terms, however, are given by equations (15b), (17b), and (24) with $\dot{\phi} = 0$.

$$\left. \begin{aligned} I(\ddot{\sigma} - \sigma \dot{\lambda}^2) &= Q S l \left[\sigma C_{m\alpha} + \frac{\dot{\sigma} l}{V} (C_{mq} + C_{m\dot{\alpha}}) + \sigma \left(\frac{\dot{\lambda} l}{V} \right)^2 C_{mPq} \right] \\ I(\ddot{\sigma} \dot{\lambda} + 2\dot{\sigma} \dot{\lambda}) &= Q S l \left[\sigma \frac{\dot{\lambda} l}{V} C_{mq} - \sigma \frac{\dot{\lambda} l}{V} C_{mP\alpha} - \frac{\dot{\sigma} l}{V} \frac{\dot{\lambda} l}{V} (C_{mPq} + C_{mP\dot{\alpha}}) \right] \end{aligned} \right\} \quad (31)$$

In both equations the third term on the right-hand side is in phase with the second inertia term, and hence can generally be neglected by comparison. In this case the equations become

$$\left. \begin{aligned} I(\ddot{\sigma} - \sigma \dot{\lambda}^2) &= QSl \left[\sigma C_{m\alpha} + \frac{\dot{\sigma}l}{V} (C_{mq} + C_{m\dot{\alpha}}) \right] \\ I(\sigma \ddot{\lambda} + 2\dot{\sigma}\dot{\lambda}) &= QSl \sigma \frac{\dot{\lambda}l}{V} (C_{mq} - C_{m_{P\alpha}}) \end{aligned} \right\} \quad (32)$$

These equations transform in α, β coordinates to

$$\left. \begin{aligned} I\ddot{\alpha} &= QSl \left[\alpha C_{m\alpha} + \frac{\dot{\alpha}l}{V} C_{mq} + \frac{\alpha l}{V} \left(\frac{\alpha \dot{\alpha} + \beta \dot{\beta}}{\alpha^2 + \beta^2} \right) C_{m\dot{\alpha}} - \frac{\beta l}{V} \left(\frac{\beta \dot{\alpha} - \alpha \dot{\beta}}{\alpha^2 + \beta^2} \right) C_{m_{P\alpha}} \right] \\ I\ddot{\beta} &= QSl \left[\beta C_{m\alpha} + \frac{\dot{\beta}l}{V} C_{mq} + \frac{\beta l}{V} \left(\frac{\beta \dot{\beta} + \alpha \dot{\alpha}}{\alpha^2 + \beta^2} \right) C_{m\dot{\alpha}} - \frac{\alpha l}{V} \left(\frac{\alpha \dot{\beta} - \beta \dot{\alpha}}{\beta^2 + \alpha^2} \right) C_{m_{P\alpha}} \right] \end{aligned} \right\} \quad (33)$$

An important point should be noted here; whereas the equations of motion as usually written are linear in α, β coordinates (eqs. (28)), the equations of motion as formulated in this report (eqs. (32) and (33)) are inherently nonlinear and no closed solution corresponding to equation (30) is possible.

A first integral of the second of equations (32) may be obtained if it is multiplied by σ :

$$\dot{\lambda} = \frac{Ce^{-kt}}{\sigma^2} \quad (34)$$

where

$$C = \sigma_1^2 \dot{\lambda}_1$$

$$k = - \frac{(C_{mq} - C_{m_{P\alpha}})}{I} \frac{QSl^2}{V}$$

Substituting equation (34) in the first of equations (32) gives

$$\ddot{\sigma} + 2\xi\omega_n\dot{\sigma} + \omega_n^2\sigma = \frac{C^2}{\sigma^3} e^{-2kt} \quad (35)$$

where ξ and ω_n carry the same definition given previously.

The stability of the motion as given by the solution of equation (35) is determined by the signs of both $C_{mq} - C_{mP\alpha}$ and $C_{mq} + C_{m\dot{\alpha}}$; if either term is positive the motion is unstable.⁴ A case of particular interest is that for which $C_{mq} = C_{mP\alpha}$ and $C_{mq} + C_{m\dot{\alpha}} < 0$. Equations (34) and (35) then become

$$\dot{\lambda} = \frac{C}{\sigma^2} \quad (36)$$

$$\ddot{\sigma} + 2\zeta\omega_n\dot{\sigma} + \omega_n^2\sigma = \frac{C^2}{\sigma^3} \quad (37)$$

Several interesting features of the motion can be deduced without actually solving equation (37):

- (1) If $C = 0$ then $\dot{\lambda} = 0$ from equation (36) and the solution of equation (37) is given by equation (30) with γ_e having the same value for both α and β . Then the motion is planar, and when plotted in α, β coordinates, appears as a line.
- (2) If $C \neq 0$, then because of the energy dissipation term $2\zeta\omega_n$ in equation (37), as $t \rightarrow \infty$, $\ddot{\sigma} \rightarrow 0$, $\sigma \rightarrow \sigma_a = \sqrt{C/\omega_n}$, $\dot{\lambda} \rightarrow \omega_n$. Since $\alpha = \sigma \sin \lambda$, $\beta = \sigma \cos \lambda$, then $\alpha \rightarrow \alpha_a = \sigma_a \sin \omega_n t$, $\beta \rightarrow \beta_a = \sigma_a \cos \omega_n t$, and the final motion plotted in α, β coordinates is circular. This motion is given by the solution (30) only with the conditions $\alpha_0 = \beta_0$, $\gamma_\alpha = \gamma_\beta \pm 90^\circ$, and the restriction that the motion is completely undamped; that is, $\zeta = 0$.
- (3) If the energy dissipation term in equation (37) is zero, equations (36) and (37) transform into equations (28) with $C_{mq} + C_{m\dot{\alpha}} = 0$. The solution is then equation (30) with zero exponential, and the resulting motion in the α, β plane, with the proper initial conditions, is elliptical. Then if the energy dissipation term in equation (37) is not zero but sufficiently small, with the proper initial conditions the solution of equations (36) and (37) must also be approximately elliptical in the α, β plane, damping finally to the circular motion given by the asymptotic values α_a and β_a .

⁴These stability criteria apply only to the case of uniform rectilinear motion being considered. In the general case, in which transverse motion of the center of gravity and changes in forward speed are permitted, the stability is influenced in addition by the vehicle's lift-curve slope and drag coefficient.

It can be seen, then, that the equations of motion, as usually formulated, yield motions dependent only on initial conditions, whereas a more correct formulation provides a mechanism by means of which the motion can degenerate to the circular state. This is true even though the aerodynamic terms are linear functions of their respective variables. These results are of particular interest because degeneration of motion to the circular state has been observed recently in ballistic range firings (see ref. 6), and attempts to explain the phenomenon (ref. 7), by means of the usual formulation of the equations, have been successful only through the inclusion of aerodynamic nonlinearities in the angles and angular velocities.

One further interesting deduction can be made from consideration of the general equations (34) and (35); that is, the general probability of precessional motion. To see this, consider first the equations as usually written, equations (29). Note that equations (29), because their form is identical to equations (32), can be expressed in the forms (34) and (35) with k replaced by $2\xi\omega_n$. This corresponds to the condition $C_{mP\alpha} = -C_{m\dot{\alpha}}$, and the solution is given by equation (30). Then since $\sigma^2 = \alpha^2 + \beta^2$,

$$\sigma^2 = e^{-2\xi\omega_n t} \left[\alpha_0^2 \sin^2(\omega t + \gamma_\alpha) + \beta_0^2 \sin^2(\omega t + \gamma_\beta) \right]$$

Substitution into equation (34) eliminates the exponential time dependency of $\dot{\lambda}$ giving

$$\dot{\lambda} = \frac{C}{\alpha_0^2 \sin^2(\omega t + \gamma_\alpha) + \beta_0^2 \sin^2(\omega t + \gamma_\beta)}$$

That is, $\dot{\lambda}$ is a periodic function of time. Integration of $\dot{\lambda}$ over a cycle of σ yields a constant, ω , for the average value of $\dot{\lambda}$ per cycle of σ . But 2ω is the circular frequency of σ , so that σ completes precisely two cycles in the time λ completes one revolution. Thus, with $\phi = 0$, the solution of the equations of motion as usually written (and with linear aerodynamics) precludes the existence of precessional motion, a well-known result.

Now consider the case $C_{mP\alpha} \neq -C_{m\dot{\alpha}}$. By analogy with the solution of a linear second-order differential equation, the period of σ obtained from the solution of equation (35) will be relatively independent of both ξ and k for most practical values of these constants. The exponential decay rate of σ will no longer cancel that for $\dot{\lambda}$ in equation (34), however, so that the average value of $\dot{\lambda}$ per cycle of σ will change from cycle to cycle, thus resulting in a motion which will precess.

Figures 2(a), 2(b), and 2(c) show motions computed from equations (34) and (35); the constants used in the computations are given in table I. In figure 2(a), the value of $C_{mP\alpha}$ is negative and equal to $C_{m\dot{\alpha}}$ and as

a result the motion degenerates to a circle as previously described. In figure 2(b), $C_{mP\alpha}$ is positive and of the same order of magnitude as C_{mq} ; note that the motion precesses slightly. In figure 2(c), $C_{mP\alpha}$ is positive and perhaps an order of magnitude larger than C_{mq} and, as can be seen, causes an initially large precession. The value of $C_{mP\alpha}$ in figure 2(c) is probably unrealistically large, however; for values of $C_{mP\alpha}$ likely to be encountered in practice the precessional motion is more likely to resemble that of figure 2(b).

Suggested Experiment

As has been mentioned, the analysis given here, while it corrects an error made in previous analyses, still contains with those analyses an unexamined assumption; namely, that the forces acting in a plane due to motion in that plane are unaffected by motion in another plane. An experiment is proposed below for investigating the validity of this assumption.

Consider first a body undergoing circular motion, and, as before, let the spin rate $\dot{\phi}$ be equal to $\dot{\lambda}$, and let $\dot{\lambda}$ be constant. Then Magnus forces are identically zero and the motion is completely steady. The normal velocities are $W/V = \sigma = \text{constant}$ in the plane of σ and $W/V = \sigma \dot{\lambda} x / V = Kx$ normal to the plane of σ , where K is a constant. These normal velocities can be simulated in the wind tunnel with a stationary model. The normal velocity in the σ plane is reproduced simply by placing the model at an angle of attack equal to σ . The normal velocity in the plane normal to the σ plane is simulated for small σ by having the model's axis of symmetry curved in that plane so that the free-stream velocity component normal to the axis varies linearly with x . Then the effect of motion in the direction normal to the σ plane on forces in the σ plane would be inferred from differences between forces in the σ plane measured with the curved model and those measured at the same angle of attack with a model having no curvature. Conversely, the effect of motion in the σ plane on forces in the plane normal to the σ plane would be inferred from differences between forces in the latter plane measured with the curved model at angle of attack σ and at zero angle of attack.

Consider next an unsteady motion. Let the body undergo a nearly circular elliptical motion in which again $\dot{\phi} = \dot{\lambda} \approx \text{constant}$. Then again Magnus forces are zero but the motion is unsteady because σ varies with time. Observe that during this motion σ varies periodically about a constant inclination. Hence the normal velocity in the plane of σ is $W/V = \sigma - \dot{\sigma} x / V$ with $\sigma = \sigma_0 + a \cos \omega t$. Normal to the plane of σ the normal velocity is $W/V \approx \sigma_0 (\dot{\lambda} x / V) \approx Kx$ so long as $\sigma_0 \gg a$. The foregoing normal velocities can again be simulated in the wind tunnel if the curved model is mounted on a dynamic balance. The normal velocity in the σ plane is reproduced by placing the model at angle of attack σ_0 and causing

it to oscillate with amplitude a . The model's curvature is in the plane normal to the plane of σ and again simulates the linear variation in x of normal velocity in that plane. Observe that to approximate an elliptical motion the body must complete two oscillations in the time λ completes one revolution. Hence, $\omega = 2\lambda$, a compatibility condition that relates the model's reduced frequency, angle of attack, and curvature. As before, the effect of motion normal to the σ plane on forces in the σ plane would be inferred from differences between oscillatory forces in the σ plane measured with the curved model and those measured at the same angle of attack, frequency, and amplitude of oscillation with a model having no curvature. The effect of unsteady motion in the σ plane on forces in the plane normal to the σ plane would be inferred from differences between forces in the latter plane measured with the curved model first oscillating in the σ plane at mean angle of attack σ_0 and then stationary at that angle of attack.

CONCLUDING REMARKS

The foregoing analysis has shown that significant errors exist in the accepted practice of writing the aerodynamic forces and moments on a body of revolution during combined pitching and yawing motion. A more correct formulation has been presented which shows that even with linear aerodynamic coefficients the differential equations governing the motion are fundamentally nonlinear. This permits an explanation of a wider class of motions than has been previously considered possible under the restriction of linear aerodynamics. In particular, the degeneration of motion to the circular state, which has been observed experimentally, is explainable under the revised theory without the necessity of introducing nonlinear aerodynamics.

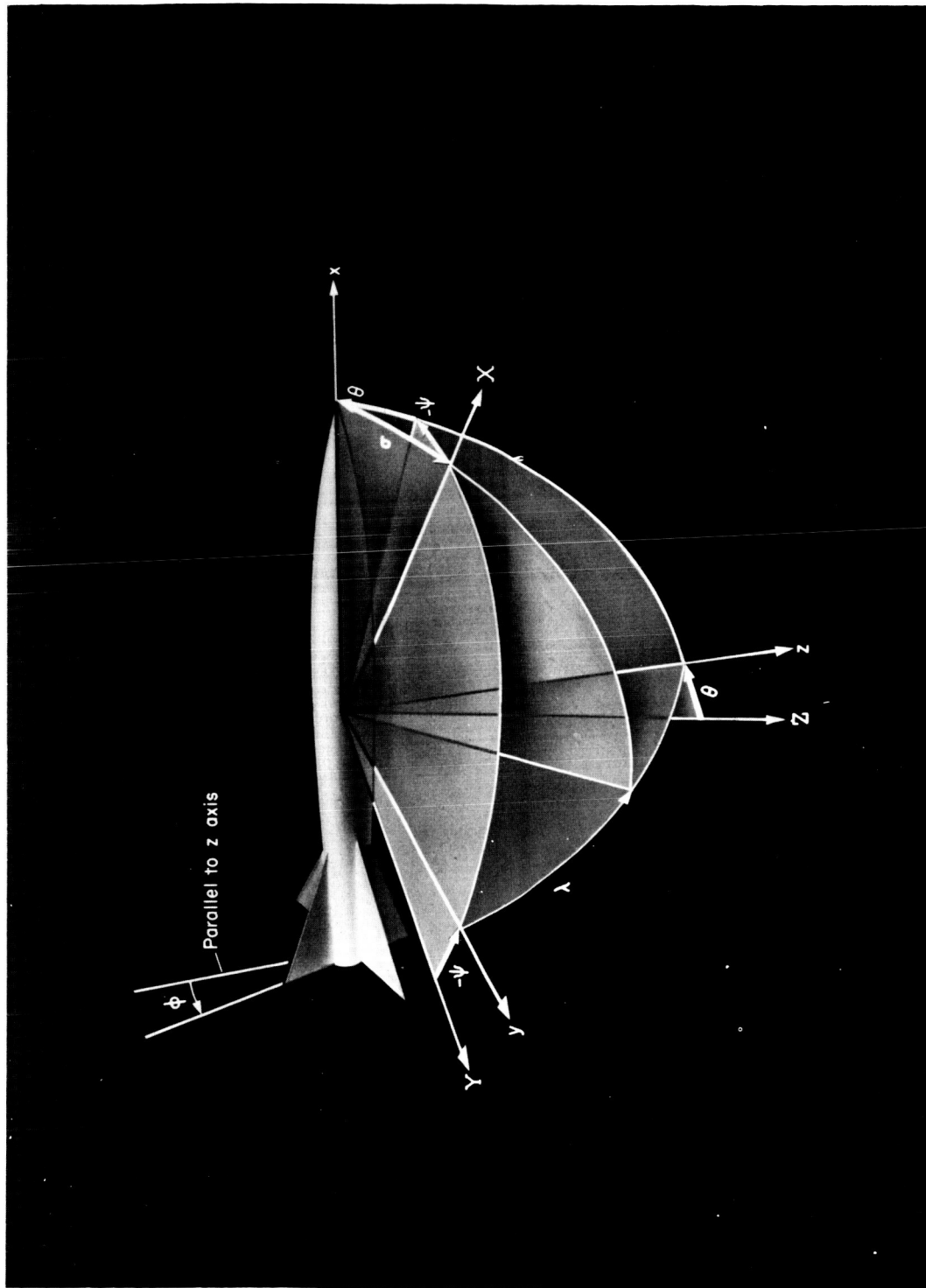
Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Jan. 14, 1960

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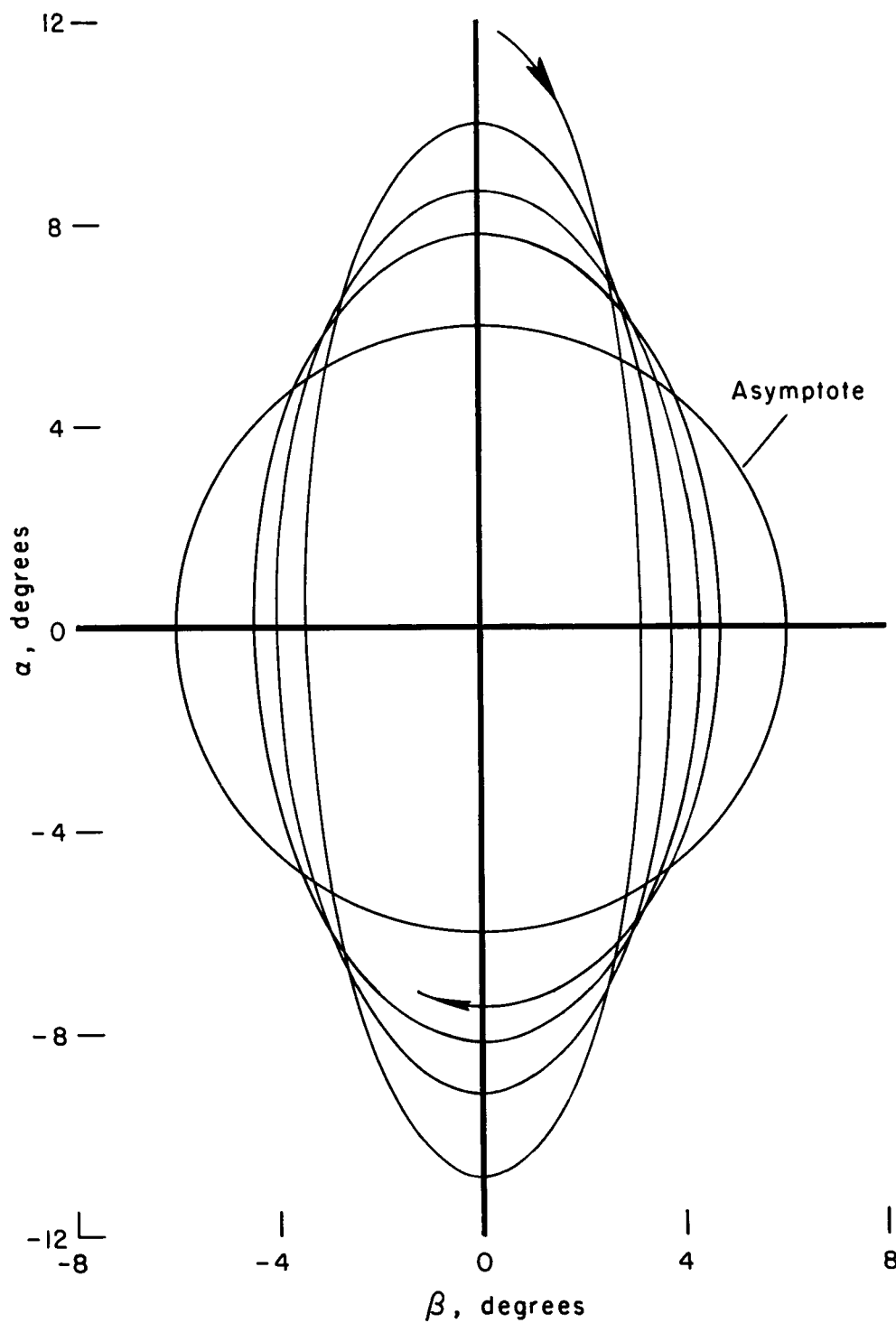
TABLE I.- CONSTANTS USED IN THE COMPUTATIONS

Figure	ζ	ω_n , per sec	k, per sec	C, per sec
2(a)	0.055	120	0	1.32
2(b)	.055	120	27.8	1.32
2(c)	.031	120	41.6	5.26



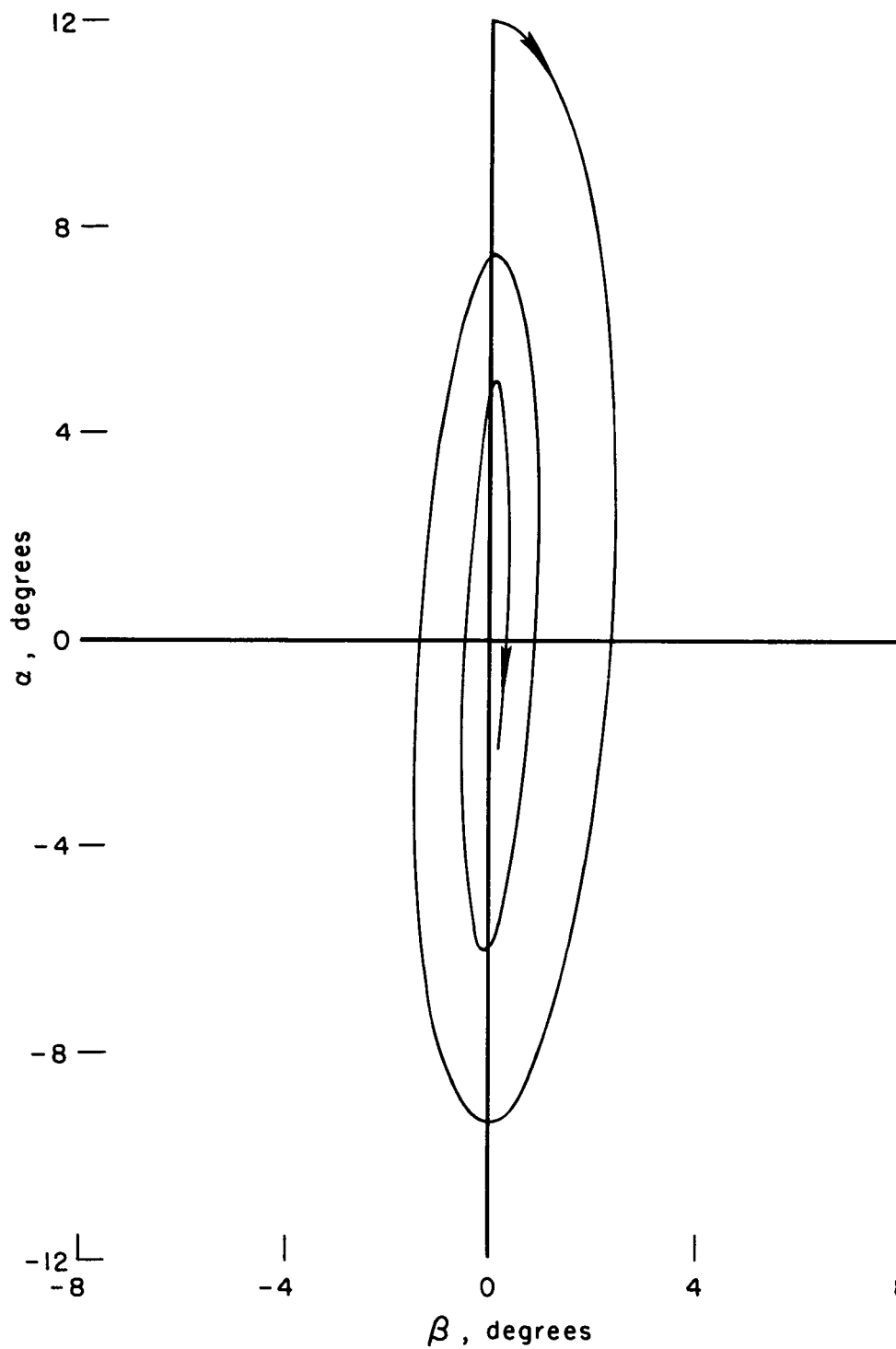
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Figure 1.- Coordinate system.



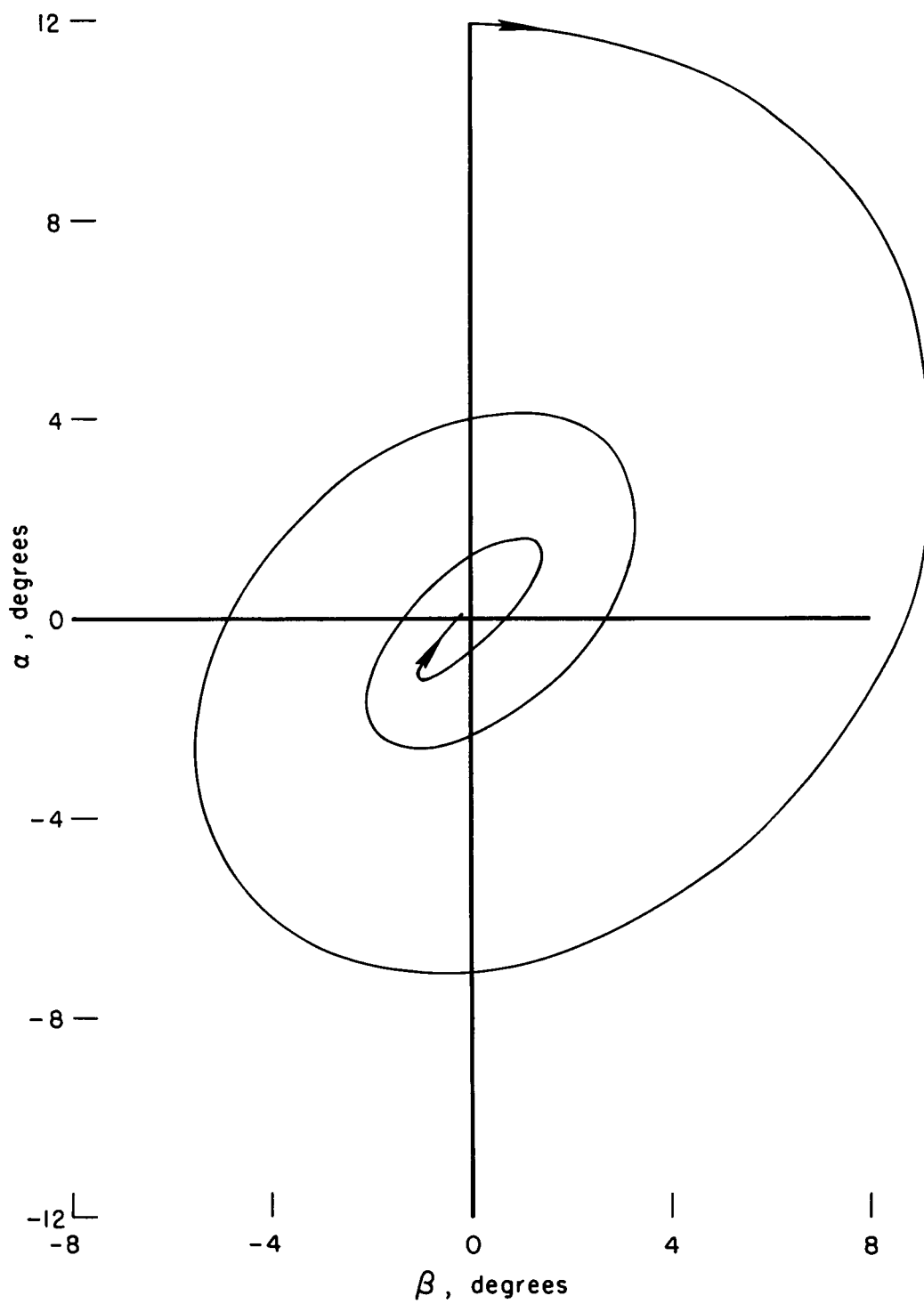
(a) $C_{mP\alpha} = C_{mq}$

Figure 2.- Pitching and yawing motion of a nonspinning nonplunging body.



(b) $C_{m_{P\alpha}} \approx -C_{m_q}$

Figure 2.- Continued.



(c) $C_{m_{P\alpha}} \approx -10 C_{m_q}$

Figure 2.- Concluded.